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NEW H-INFINITY TRACKING CONTROL ALGORITHM BASED ON INTEGRAL EQUATION APPROACH IN LINEAR CONTINUOUS-TIME SYSTEMS

Seiichi Nakamori

Department of Technical Education, Kagoshima University
1-20-6, Korimoto, Kagoshima, Japan.
E-mail: k6165161@kadai.jp

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ABSTRACT

In linear continuous-time systems with the control and exogenous inputs, this paper proposes a new H-infinity tracking control algorithm.In this study, we extended the existing quadratic tracking control method to the H-infinity tracking control method. The variational method gives the integral equation of the second kind as a necessary condition of the control law for the quadratic performance function regarding the H-infinity tracking control problem. The H-infinity tracking control algorithm differs from other H-infinity tracking control algorithms in that it is derived using the integral equation of the second kind. In Theorem 3 the proposed H-infinity tracking control algorithm is designed in conjunction with the Luenberger state observer. In Theorem 2 the Luenberger state observer, which is based on linear matrix inequalities (LMIs), is shown as an example in state estimation. In Theorem 1 it is shown that the integral equation of the second kind for the tracking control problem is transformed into the Euler-Lagrange equation, where the equations for the control and exogenous inputs are given.

A numerical simulation example demonstrates the feasibility of the H-infinity tracking control algorithm presented in this research for linear continuous-time system. Here, it is noteworthy that the exogenous input is additional to the control input. The output of the system approaches the desired value asymptotically as time passes.

Keywords: H-infinity tracking control, Control input, Exogenous input, Linear matrix inequalities, Euler-Lagrange equation.

Mathematics Subject Classification: 49N10, 49N70.

1. INTRODUCTION

The H-infinity control problems for the regulator (Gadewadikar *et al.*, 2006; Duan and Yu, 2013) and tracking control (Modares *et al.*, 2015) have been thoroughly investigated. The quadratic regulator problem is addressed by Kailath (1972). Based on the variational principle, Kailath (1972) obtains an integral equation of the second kind as a necessary condition for the control law by substituting the solution of the state differential equation into the quadratic cost function. Nakamori and Hataji (1982) propose the quadratic tracking algorithm by deriving the integral equation of the second kind to the tracking problem, employing the technique of Kailath (1972). In Moghadam and Lewis (2019), a sub-optimal output-feedback H-infinity control law for tracking in linear partially unknown linear continuous-time systems is designed using an online integral reinforcement learning-based algorithm. Zhang *et al.* (2018) propose a neural network (NN)-based online model-free integral reinforcement learning algorithm to solve the finite-horizon optimal tracking control problem for completely unknown nonlinear continuous-time systems with disturbance and saturating actuators. Nakamori (2021) has developed a proportional-integral-differential (PID) control technique that uses Arduino to keep a closed space at a constant temperature by pulse width modulation (PWM) control of an alternating current (AC) voltage of 100 [V]. The PID controller does not require state space model information about controlled objects. Zheng *et al.* (2002) and He and Wang (2006), based on the iterative linear matrix inequality (ILMI) technique, and Shimizu (2016) and Goyal *et al.* (2020), based on the (LMIs), propose multivariate H-infinity PID controllers, assuming knowledge of a state-space model of the controlled object.

This study proposes a new H-infinity tracking control algorithm in Theorem 3 for linear continuous-time systems as an extension of the quadratic tracking controller (Nakamori and Hataji, 1982). In the linear system with the control input $u_1(t)$ and the exogenous input $u_2(t)$, this work introduces the quadratic performance function in the H-infinity tracking control problem. We derive the integral equation of the second kind as a necessary condition for the control law in the H-infinity tracking control problem using variational calculus in the H-infinity quadratic performance function. From the integral equation the Euler-Lagrange equation is obtained in Theorem 1. Here, the equations for the control and exogenous inputs are given. The proposed H-infinity tracking control algorithm in Theorem 3 differs from existing algorithms in that the algorithm is derived from the integral equation. The estimates of the control and exogenous inputs use the estimates of the state variables in the H-infinity tracking control algorithm combined with the Luenberger state observer. The state variables are estimated by the Luenberger state observer. For example, in Theorem 2, the observer gain is determined using LMIs based on Lyapunov stability theory (Duan and Yu, 2013). Refer to Ogata (1996) and Rossiter (1998) for the pole placement method of calculating observer gain.

The H-infinity tracking control problem is introduced in Section 2.The H-infinity tracking control algorithm is proposed in Section 3 in combination with the LMIs-based state observer. A numerical simulation example in section 4 demonstrates the feasibility of the H-infinity tracking control algorithm presented in this research.

2. H-INFINITY TRACKING CONTROL PROBLEM

Let the state-space model for the state $x(t)$ and the output equation for the output $z(t)$ be given by

$$\frac{dx(t)}{dt} = A(t)x(t) + Gu(t),$$
$$G = [G_1 G_2],\ u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$
$$z(t) = D(t)x(t), \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input, and $z(t) \in R^l$ is the output, respectively. $u_1(t) \in R^{m_1}$ is the control input and $u_2(t) \in R^{m_2}$, $m_1 + m_2 = m$, is the exogenous input. We consider an H-infinity tracking control problem, where the output tracks the target function $\eta(t)$. Let us define $\|\tilde{z}(t)\|^2_{L_2[0,T]}$ as described in (2). $\tilde{z}(t)$ is referred to as a performance output (Gadewadikar *et al.*, 2006).

$$\|\tilde{z}(t)\|^2_{L_2[0,T]} = (\eta(t) - z((t))^T Q(t)(\eta(t) - z(t)) + u_1^T(t)\tilde{R}(t)u_1(t),\ 0 \le t \le T \tag{2}$$

We consider the standard H-infinity tracking control problem. The system $L_2[0, T]$ gain is bounded or attenuated by γ

$$\sup_{u_2 \in L_2[0,T]} \frac{\|\tilde{z}(t)\|_{L_2[0,T]}}{\|u_2(t)\|_{L_2[0,T]}} < \gamma, \tag{3}$$

where $\gamma > 0$ denotes a constant. The problem of finding the minimum value of γ such that the H-infinity control problem is solvable is called the H-infinity optimization problem. Clearly, this problem can also be described as (Bhattacharyya *et al.*, 2009)

$$\|\tilde{z}(t)\|^2_{L_2[0,T]} < \gamma^2 \|u_2(t)\|^2_{L_2[0,T]}, \ \forall u_2(t). \tag{4}$$

Let's modify the quadratic performance function from the infinite H-infinity regulator problem (Gadewadikar *et al.*, 2006) to the finite-horizon H-infinity tracking control problem as follows.

$$J(x, u_1, u_2)$$
$$= \int_0^T [(\eta(t)-z(t))^T Q(t)(\eta(t)-z(t)) + u_1^T(t)\tilde{R}(t)u_1(t) - \gamma^2 u_2^T(t)u_2(t)]dt \tag{5}$$

Given γ^2, we investigate minimizing $J(x, u_1, u_2)$ with regard to $u_1(t)$ and maximizing $J(x, u_1, u_2)$ with respect to $u_2(t)$.

$$\frac{\int_0^T \|\tilde{z}(t)\|^2_{L_2[0,T]} dt}{\int_0^T \|u_2(t)\|^2_{L_2[0,T]} dt}$$

$$\frac{\int_0^T (\eta(t)-z(t))^T Q(t)(\eta(t)-z(t))dt + \int_0^T u_1^T(t)\tilde{R}(t)u_1(t)dy}{\int_0^T u_2^T(t)u_2(t)dt} \le \gamma^2. \tag{6}$$

The worst-case disturbance, $u_2(t)$, is the exogenous input, whereas $u_1(t)$ is the control input.Introducing $R(t) = \begin{bmatrix} \tilde{R}(t) & 0 \\ 0 & -\gamma^2 I_{m_2 \times m_2} \end{bmatrix}$, we can express (5) as follows.

$$J(x, u_1, u_2) = \int_0^T [(\eta(t)-z(t))^T Q(t)(\eta(t)-z(t)) + u^T(t)R(t)u(t)]dt$$
$$= \int_0^T [\|\eta(t)-z(t)\|^2_{Q(t)} + \|u(t)\|^2_{R(t)}]dt \tag{7}$$

(7) represents the quadratic performance function for the H-infinity tracking control problem. Given γ^2, the H-infinity tracking problem is reduced to the minimization of the quadratic tracking performance (7) with respect to $u(t)\cdot x(t)$ is expressed as

$$x(t) = \Phi(t,0)c + \int_0^T \Phi(t,s)Gu(s)ds, \ \Phi(t,s) = e^{A(t-s)},$$
$$1(\alpha) = \begin{cases} 1, & 0 \le \alpha, \\ 0, & \alpha < 0, \end{cases}$$
$$x(t) = \Phi(t,0)c + \int_0^T 1(t-s)\Phi(t,s)Gu(s)ds, \ 0 \leqq s, \ t \leqq T < \infty. \tag{8}$$

Here, $\Phi(t, s)$ is the state-transition matrix. Substituting (8) into (7), we have

$$J(x,u_1,u_2)=\int_0^T[\|\eta(t)-D(t)\Phi(t,0)c-D(t)\int_0^T 1(t-s)\Phi(t,s)Gu(s)ds\|_{Q(t)}^2$$
$$+\|u(t)\|_{R(t)}^2\,dt. \tag{9}$$

In order to minimize the quadratic performance function (9), we use the variational principle. Let $\hat{u}(t)$ be quantity that minimizes (9). Let $\hat{\gamma}(t)$ be a continuously differentiable function in the interval $0 \le t \le T$ and ε be a scalar parameter with an extremely small positive value. Putting

$$u(t)=\hat{u}(t)+\varepsilon\hat{\gamma}(t), \tag{10}$$

we substitute (10) into (9). In the calculation of the variation $\Delta J = J(\hat{u}(t)) - J(u(t))$, the term ε is the first variation, while the term ε^2 is the second variation. Putting the first variation to zero and replacing $u(t)$ with $\hat{u}(t)$, we obtain the necessary condition for the minimum of $J(x, u_1, u_2)$ as follows.

$$R(t)(\hat{u}(t)+\int_0^T\int_0^T 1(s-t)1(s-\tau)G^T\Phi^T(s,t)D^T(s)Q(s)D(s,\tau)G\hat{u}(\tau)d\tau ds$$
$$=\int_0^T 1(s-\tau)G^T\Phi^T(s,t)D^T(s)Q(s)\,(\eta(s)-D(s,0)c)ds \tag{11}$$

Introducing

$$K(t,s)=\begin{cases}\int_t^T G^T\Phi^T(\tau,t)D^T(\tau)Q(\tau)D(\tau)\Phi(\tau,s)d\tau,\ 0\leqq s\leqq t\leqq T,\\ \int_s^T G^T\Phi^T(\tau,t)D^T(\tau)Q(\tau)D(\tau)\Phi(\tau,s)d\tau,\ 0\leqq t\leqq s\leqq T,\end{cases} \tag{12}$$

and

$$m(t)=-\int_t^T G^T\Phi^T(\tau,t)D^T(\tau)Q(\tau)D(\tau)\Phi(\tau,0)c-\eta(\tau))d\tau, \tag{13}$$

we have an integral equation

$$R(t)u(t)+\int_0^T K(t,\tau)G\hat{u}(\tau)d\tau=m(t). \tag{14}$$

From the property that the second variation is positive, we get

$$R(t)\delta(t-s)+K(t,s)G>0,0\leqq s,t\leqq T<\infty. \tag{15}$$

(15) is a sufficient condition for the min-max of the performance criterion $J(x, u_1, u_2)$. $\delta(t-s)$ denotes the Dirac delta function. Kailath (1972), in the quadratic regulator problem, proposes the equations corresponding to (12)-(14). (12)-(14) show a natural extension of the quadratic regulator problem to the H-infinity tracking control problem through the quadratic tracking problem (Nakamori and Hataji, 1982). Under the formulation of the H-infinity tracking control problem above, in section 3, from the integral equation (14), a new H-infinity tracking control algorithm is presented.

3. H-INFINITY TRACKING CONTROL ALGORITHM AND STATE OBSERVER

Theorem 1 proposes the Euler-Lagrange equation in the H-infinity tracking control problem. A two-point boundary value problem (TPBVP), for the control input $u_1(t)$ and the exogenous input $u_2(t)$, is shown.

Theorem 1

The integral equation (14), obtained in section 2 for the H-infinity tracking control problem, is transformed into the Euler-Lagrange equation in the linear continuous-time system (1). The TPBVP consists of (16) and (17). In the H-infinity tracking control, the control input $u_1(t)$ and the exogenous input $u_2(t)$ are given by (18) and (19), respectively.

$$\frac{d\lambda(t)}{dt} = -2D^T(t)Q(t)(D(t)x(t) - \eta(t)) - A^T\lambda(t),$$
$$\lambda(T) = 0 \text{ (terminal condition)) } \tag{16}$$

$$\frac{dx(t)}{dt} = Ax(t) + Gu(t) = Ax(t) + G_1u_1(t) + G_2u_2(t),$$
$$x(0) = c \text{ (initial condition)} \tag{17}$$

Control input: $u_1(t)$

$$u_1(t) = -0.5\tilde{R}^{-1}(t)G_1^T\lambda(t) \tag{18}$$

Exogenous input: $u_2(t)$

$$u_2(t) = 0.5\gamma^{-2}G_2^T\lambda(t) \tag{19}$$

Proof of Theorem 1 is deferred to Appendix A.

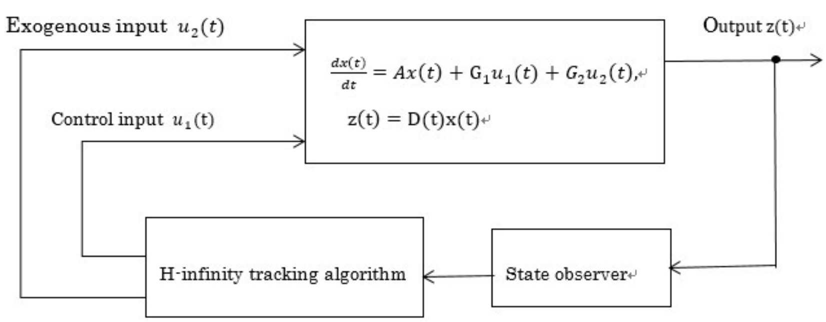

Fig. 1: Combination of H-infinity tracking control with state observer.

Fig. 1 illustrates the block diagram of the H-infinity tracking control combined with the state observer. Theorem 2 shows the Luenberger state observer by the LMIs (Duan and Yu, 2013) in estimating the state $x(t)$ using the output $z(t)$.

Theorem 2

For the system (1), (20) shows the Luenberger state observer.

$$\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + G\hat{u}(t) + L(C\hat{x}(t) - z(t)), \quad \hat{x}(0) = 0, \quad \hat{u}(t) = \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix}, \tag{20}$$

where the estimate $\hat{u}(t)$ of the input $u(t)$ consists of the estimate $\hat{u}_1(t)$ of the control input $u_1(t)$ and the estimate $\hat{u}_2(t)$ of the exogenous input $u_2(t)$. The state observer gain lin (20) is calculated by the LMIs (Duan and Yu, 2013).

$$PA + A^T P + WC + C^T W^T < 0, P > 0, L = P^{-1} W, \tag{21}$$

from the asymptotic stability of

$$\frac{de(t)}{dt} = (A + LC)e(t), \; e(t) = x(t) - \hat{x}(t), \tag{22}$$

based on the Lyapunov stability theory.

Theorem 3 proposes the H-infinity control algorithm for the estimates of the control input $u_1(t)$ and the exogenous input $u_2(t)$ from the integral equation (14).

Theorem 3

For $u(t)$ with the components of the control input $u_1(t)$ and the exogenous input $u_2(t)$,

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \tag{23}$$

the estimate $\hat{u}(t)$ of $u(t)$ is calculated by (24)-(27).

$$\hat{u}(t) = \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix}, \; \hat{u}(t) = R^{-1}(t) G^T \overline{P} \hat{x}(t) + R^{-1}(t) G^T \overline{\xi} \tag{24}$$

$$\frac{dP(t)}{dt} = -A^T P(t) - P(t)A(t) - P(t)GR^{-1}(t)G^T P(t) + D^T(t)Q(t)D(t), \; P(T) = 0 \tag{25}$$

$$\frac{d\xi(t)}{dt} = -(A^T + P(t)GR^{-1}(t)G^T)\xi(t) - D^T(t)Q(t)\eta(t), \; \xi(T) = 0 \tag{26}$$

$$R(t) = \begin{bmatrix} \widetilde{R}(t) & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \tag{27}$$

In (24) we use the estimate $\hat{x}(t)$ of the state $x(t)$, calculated by the Luenberger state observer of Theorem 2, instead of $x(t)$. $P(t)$ and $\xi(t)$ are computed, starting from time $t = T$, in the reverse direction of time, until they arrive at stationary values $\overline{P}$ and $\overline{\xi}$, respectively. The estimate $\hat{u}(t)$ of $u(t)$ is calculated by (24) using $\overline{P}$ and $\overline{\xi}$.

Proof of Theorem 3 is deferred to Appendix B.

Section 4 presents a numerical simulation example to demonstrate the efficiency of the proposed H-infinity tracking control algorithm.

4. A NUMERICAL SIMULATION EXAMPLE

Let the state differential equation for the state $x(t)$ and the output equation for the output $z(t)$ be given by

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \; z(t) = [1 \;\; 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{28}$$

The system of (28) satisfies the controllability and observability conditions. It is worth noting that the exogenous input $u_2(t)$ is additional to the control input $u_1(t)$. By substituting the system matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, the input matrix $G = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, the observation vector $D(t) = [1\ 0]$, $R(t) = \begin{bmatrix} \tilde{R}(t) & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, $\tilde{R}(t) = 1$, for $\gamma = 10$ or 1.1, and the desired value $\eta = 10$ into the H-infinity tracking control algorithm of Theorem 3 and the state observer of Theorem 2, the estimate $\hat{x}(t)$, $\hat{x}(k) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}$, of the state $x(t)$ and the estimate $\hat{u}(t)$, $\hat{u}(t) = \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix}$, of $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$, are calculated. In (24) the stationary values $\overline{P}$ and $\overline{\xi}$ are calculated as $\overline{P} = \begin{bmatrix} -1.4178 & -1.0050 \\ -1.0050 & -1.4249 \end{bmatrix}$, $\overline{\xi} = \begin{bmatrix} 14.1777 \\ 10.0503 \end{bmatrix}$ for $\gamma = 10$. For the state observer gain $L = \begin{bmatrix} -0.8750 \\ -1.2917 \end{bmatrix}$, which is computed using the LMIs of Theorem 2, the eigenvalues of $A + LC$ in the Luenberger state observer in (20) are $-0.4375 \pm 1.04893i$. Because the real part of the eigenvalues is negative, the state observer is stable. Fig. 2 illustrates the estimate $\hat{x}_1(t)$ of the state variable $x_1(t)$ vs. time t. Starting with 0, $\hat{x}_1(t)$ for $\gamma = 10$ asymptotically approaches the desired value of $\eta = 10$ as time advances. At time $t = 10$, there is still a deviation from the desired value 10 in the case of $\hat{x}_1(t)$ for $\gamma = 1.1$. Fig. 2 shows that $\hat{x}_1(t)$ for $\gamma = 10$ is feasible from short peak time and transient property compared to the case r=1.1.For $\gamma = 1.1$, at time $t = 10$, $\hat{x}_1(t)$ has a residual deviation from the desired value of 10. Fig. 3 illustrates the estimate $\hat{x}_2(t)$ of the state variable $x_2(t)$ vs. time t. For $\gamma = 10$, $\hat{x}_2(t)$ approaches 0 as time passes. At time $t = 10$, $\hat{x}_2(t)$ for $\gamma = 1.1$ does not converge to oprecisely. $\hat{x}_2(t)$ for $\gamma = 10$ shows a large fluctuation around the interval $0 \leq t \leq 5$ when compared to the case for $\gamma = 1.1$. Fig. 4 illustrates the estimate $\hat{u}_1(t)$ of the control input $u_1(t)$ vs. time t for $\gamma = 1.1$ and $\gamma = 10$. As time passes, the control input $\hat{u}_1(t)$ approaches 0. Around the interval $0 \leq t \leq 8$, the fluctuation of the estimate $\hat{u}_1(t)$ for $\gamma = 1.1$ is larger than that of $\gamma = 10$. Fig. 5 illustrates the estimate $\hat{u}_2(t)$ of the exogenous input $u_2(t)$ vs. time t for $\gamma = 1.1$ and $\gamma = 10$. As time passes, $\hat{u}_2(t)$ for $\gamma = 1.1$ approaches 0. Around the interval $0 \leq t \leq 8$, the fluctuation of $\hat{u}_2(t)$ for $\gamma = 1.1$ is large. For $0 \leq t \leq 10$, $\hat{u}_2(t)$ for $\gamma = 10$ has a value of almost 0 all of the time. The H-infinity tracking control algorithm diverges especially for small values of γ, such as $\gamma \leq 1$. The proposed H-infinity tracking algorithm, on the other hand, is reduced to the quadratic tracking control algorithm for an infinite value of γ.

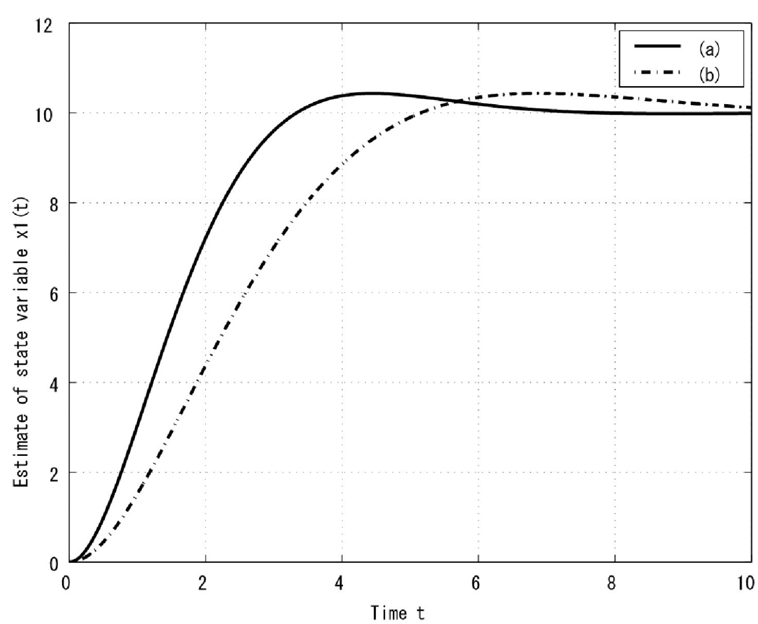

Fig. 2: Estimate $\hat{x}_1(t)$ of state variable $x_1(t)$ vs. t for (a) $\gamma = 10$ and (b) $\gamma = 1.1$.

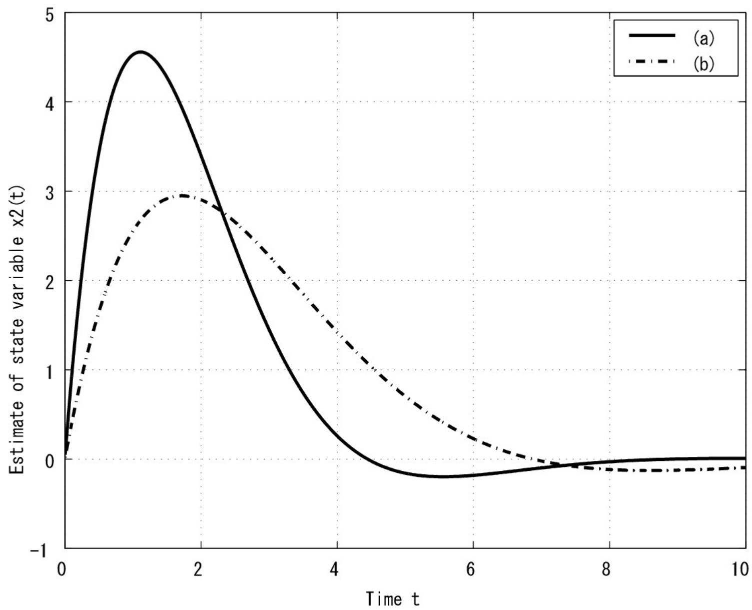

Fig. 3: Estimate $\hat{x}_2(t)$ of state variable $x_2(t)$ vs. t for (a) $\gamma = 10$ and (b) $\gamma = 1.1$.

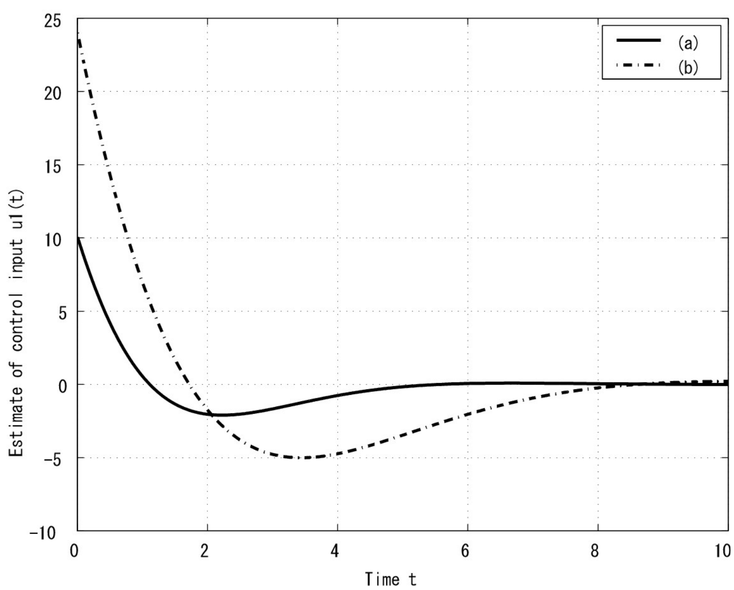

Fig. 4: Estimate $\hat{u}_1(t)$ of control input $u_1(t)$ vs. t for (a) $\gamma = 10$ and (b) $\gamma = 1.1$.

In the computation of the numerical integration, the fourth-order Runge-Kutta Gill method with a sampling interval 0.005 is used. The

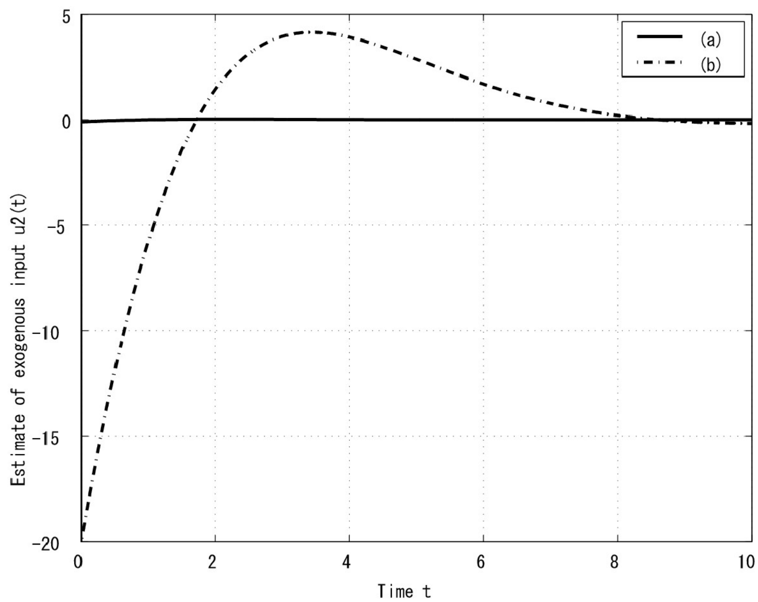

Fig. 5: Estimate $\hat{u}_2(t)$ of exogenous input $u_2(t)$ vs. t for (a) $\gamma = 10$ and (b) $\gamma = 1.1$.

program in the simulation is implemented by Octave with the exception of the figures produced by MATLAB R12. In the computation of LMIs, YALMIP is used.

5. CONCLUSIONS

This paper has proposed the H-infinity tracking control algorithm combined with the Luenberger state observer. In Theorem 2, the LMIs calculate the observer gain. In the numerical simulation example, starting at time 0, the estimate $\hat{x}_1(t)$ of the state variable $x_1(t)$ approaches the desired value of $\eta = 10$ asymptotically for $\gamma = 10$ as time passes. Fig. 2 shows that $\hat{x}_1(t)$ for $\gamma = 10$ is feasible from short peak time and transient property compared to the case $\gamma = 1.1$. The H-infinity tracking control algorithm diverges especially for small values of γ, such as $\gamma \le 1$. The proposed H-infinity tracking algorithm, on the other hand, is reduced to the quadratic tracking control algorithm for an infinite value of γ.

Main feature of this paper is that the H-infinity tracking control algorithm is derived from the integral equation of the second kind.

Appendix A: Proof of Theorem 1

By introducing $n \times n$ matrix B, which satisfies (A-1), and using (12), (13), (A-2) and (A-3), (14) is transformed into (A-4).

$$R(t) = G^T B(t) G \tag{A-1}$$

$$\widetilde{K}(t,s) = \begin{cases} \widetilde{K}_1(t,s), 0 \leqq s \leqq t \leqq T, \\ \widetilde{K}_2(t,s), 0 \leqq t \leqq s \leqq T \end{cases}$$

$$= \begin{cases} \int_t^T \Phi^T(\tau,t) D^T(\tau) Q(\tau) D(\tau) \Phi(\tau,s) d\tau, \\ 0 \leqq s \leqq t \leqq T, \\ \int_s^T \Phi^T(\tau,t) D^T(\tau) Q(\tau) D(\tau) \Phi(\tau,s) d\tau, \\ 0 \leqq t \leqq s \leqq T \end{cases} \tag{A-2}$$

$$\widetilde{m(t)} = -\int_t^T \Phi^T(\tau,t) D^T(\tau) Q(\tau)(D(\tau)\Phi(\tau,0)c - \eta(\tau)) d\tau \tag{A-3}$$

$$G^T(B(t)G\hat{u}(t) + \int_0^t \widetilde{K}_1(t,\tau)G\hat{u}(\tau)d\tau + \int_t^T \widetilde{K}_2(t,\tau)G\hat{u}(\tau)d\tau) = G^T \widetilde{m}(t) \tag{A-4}$$

The sufficient condition for (A-4) to hold is given by

$$(B(t)G\hat{u}(t) + \int_0^t \widetilde{K}_1(t,\tau)G\hat{u}(\tau)d\tau + \int_t^T \widetilde{K}_2(t,\tau)G\hat{u}(\tau)d\tau) = \widetilde{m}(t). \tag{A-5}$$

Introducing

$$\alpha(t) = (B(t)G\hat{u}(t) \tag{A-6}$$

and differentiating (A-5) with respect to t, we have

$$\frac{d\alpha(t)}{dt} + \int_0^t \widetilde{K}_{1t}(t,\tau)G\hat{u}(\tau)d\tau + \int_t^T \widetilde{K}_{2t}(t,\tau)G\hat{u}(\tau)d\tau) = \frac{d\widetilde{m}(t)}{dt}. \tag{A-7}$$

Here, $\widetilde{K}_{1t}(t,\tau) = \dfrac{\partial \widetilde{K}_1(t,\tau)}{\partial t}, \widetilde{K}_{2t}(t,\tau) = \dfrac{\partial \widetilde{K}_2(t,\tau)}{\partial t}$. From (A-2), we have

$$\frac{\partial \widetilde{K}_1(t,\tau)}{\partial t} = -D^T(t)Q(t)D(t)\Phi(t,\tau) - A^T \widetilde{K}_1(t,\tau), \tag{A-8}$$

$$\frac{\partial \widetilde{K}_2(t,\tau)}{\partial t} = -A^T \widetilde{K}_1(t,\tau). \tag{A-9}$$

From (A-2), (A-8), and (A-9), (A-7) is written as

$$\frac{d\alpha(t)}{dt} - A^T \int_0^T \widetilde{K}(t,\tau)G\hat{u}(\tau)d\tau - D^T(t)Q(t)D(t)\int_0^t \Phi(t,\tau)G\hat{u}(\tau)d\tau$$

$$= \frac{d\widetilde{m}(t)}{dt}. \tag{A-10}$$

From (A-2), (A-5) and (A-6), the second term on the left-hand side of (A-10) is equal to

$$-A^T(\widetilde{m}(t) - \alpha(t)). \tag{A-11}$$

Substitution (A-11) into (A-10) yields

$$\frac{d\alpha(t)}{dt}=\frac{d\widetilde{m}(t)}{dt}+D^T(t)Q(t)D(t)\int_0^t \Phi(t,\tau)G\hat{u}(\tau)d\tau + A^T(\widetilde{m}(t)-\alpha(t)). \quad \text{(A-12)}$$

Putting

$$M(t)=D^T(t)Q(t)D(t)\int_0^t \Phi(s,\tau)G\hat{u}(\tau)d\tau,$$

$$\widetilde{M}(t)=\int_0^t \Phi(t,\tau)G\hat{u}(\tau)d\tau, \quad \text{(A-13)}$$

we have

$$M(t) = D^T(t)Q(t)D(t)\widetilde{M}(t). \quad \text{(A-14)}$$

Differentiating $\widetilde{M}(t)$ with respect to t, we have

$$\frac{d\widetilde{M}(t)}{dt}=A\widetilde{M}(t)+G\hat{u}(t). \quad \text{(A-15)}$$

Here, from (A-13), the initial condition of the differential equation (A-15) for $\widetilde{M}(t)$ at $t=0$ is $\widetilde{M}(0)=0$. Also, from (A-3), $\frac{d\widetilde{m}(t)}{dt}$ in (A-12) is expressed as

$$\frac{d\tilde{m}(t)}{dt}=-A^T\widetilde{m}(t) + D^T(t)Q(t)(D(t)\Phi(t, 0)c - \eta(t)). \quad \text{(A-16)}$$

From (A-13) and (A-16), we rewrite (A-12) as

$$\frac{d\alpha(t)}{dt}= D^T(t)Q(t)(D(t)\Phi(t, 0)c - \eta(t))$$

$$+ D^T(t)Q(t)D(t)\widetilde{M}(t) - A^T\alpha(t). \quad \text{(A-17)}$$

$\widetilde{M}(t)$ equals $x(t)$ for $x(0) = 0$. Hence, (A-17) is transformed into

$$\frac{d\alpha(t)}{dt}= D^T(t)Q(t)(D(t)x(t) - \eta(t)) - A^T \alpha(t).) \quad \text{(A-18)}$$

To compare the above result with the Euler-Lagrange equation (Sage and White, 1977) in a linear servomechanism problem, let us multiply –2 on both sides of (A-18).

$$-2\frac{d\alpha(t)}{dt}= -2D^T(t)Q(t)(D(t)x(t) - \eta(t)) + 2A^T \alpha(t)) \quad \text{(A-19)}$$

By putting

$$-2\alpha(t) = \lambda(t), \quad \text{(A-20)}$$

(A-19) becomes

$$\frac{d\lambda(t)}{dt} = -2D^T(t)Q(t)(D(t)x(t) - \eta(t)) - A^T\lambda(t). \qquad \text{(A-21)}$$

From (A-2), (A-3), (A-5), (A-6), and (A-20), the terminal condition on $\lambda(t)$ at $t = T$ is

$$\lambda(T) = 0. \qquad \text{(A-22)}$$

Also, from (A-1), (A-6) and (A-20), we have an expression for $\hat{u}(t)$.

$$\hat{u}(t) = -0.5R^{-1}(t)G^T\lambda(t) \qquad \text{(A-23)}$$

(Q.E.D.)

Appendix B: Proof of Theorem 3

$\alpha(t) = B(t)G\hat{u}(t)$ satisfies the differential equation (A-18). From (A-1) and (A-6), $\hat{u}(t)$ is given by

$$\hat{u}(t) = R^{-1}(t)G^T\alpha(t). \qquad \text{(B-1)}$$

Let $\alpha(t)$ be expressed as

$$\alpha(t) = P(t)x(t) + \xi(t). \qquad \text{(B-2)}$$

Then, by differentiating (B-2) with respect to t, from (17) and (B-2), we have

$$\frac{d\alpha(t)}{dt} = \left(\frac{dP(t)}{dt} + P(t)A + P(t)GR^{-1}(t)G^TP(t)\right)x(t)$$
$$+P(t)GR^{-1}(t)G^T\xi(t) + \frac{d\xi(t)}{dt}. \qquad \text{(B-3)}$$

Substituting (B-2) into (A-18), we have

$$\frac{d\alpha(t)}{dt} = (D^T(t)Q(t)D(t) - A^TP(t))x(t) - D^T(t)Q(t)\eta(t) - A^T\xi(t). \qquad \text{(B-4)}$$

Equating (B-3) with (B-4)) and comparing the term of $x(t)$ and the other term, with each other, we get (25) and (26). $\hat{u}(t)$ is obtained by substituting (B-2) into (B-1) as

$$\hat{u}(t) = R^{-1}(t)G^T(P(t)x(t) + \xi(t)). \qquad \text{(B-5)}$$

The terminal conditions of $P(t)$ and $\xi(t)$ at $t = T$ are given by

$$P(T) = 0,\ \xi(T) = 0. \qquad \text{(B-6)}$$

In (A-5), for $t = T$, the third term of the left side is a zero vector. From (A-2) $\widetilde{K}_1(T,s)$ for $0 \le s \le t \le T$, is a zero matrix. Also, from (A-3), $\widetilde{m}(T)$ =0 is clear.

Therefore, $\hat{u}(T) = 0$ holds. From this (B-6) is obtained. In the current approach, we replace $x(t)$ with its estimate $\hat{x}(t)$. (Q.E.D.)

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